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# THE LOCAL BOUNDEDNESS OF THE PERTURBED MOTIONS OF AN IMPERFECT GYROSCOPE IN GIMBALS WITH DISSIPATIVE AND ACCELERATING FORCES†

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An unbalanced dynamically symmetrical gyroscope in gimbals with constructive imperfections is considered in a central Newtonian field of forces. It is assumed that there is a moment of forces of viscous friction acting on the axis of rotation of one of the rings of the suspension and an accelerating (electromagnetic) moment applied to the axis of rotation of another ring. The equations of motion have a partial solution for which the basic plane of the frame is perpendicular to the direction from the specified fixed point of the frame to the centre of gravitation, the basic plane of the mantle is parallel to this direction and the rotor rotates with an arbitrary constant angular velocity.

The equations of perturbed motions of the reduced system with two degrees of freedom are obtained to within third-order terms at the corresponding position of equilibrium. In the domain of admissible values of the parameters $F_0$ the characteristic equation of the system is considered and its coefficients are written down. A domain in $F_0$ is specified in which complex conjugate pairs of the eigenvalues have small moduli of the real parts but the absolute values of the second- to fourth-order off-resonance mistuning between the imaginary parts are not small. For an imperfect gyroscope in gimbals with dissipative and accelerating forces the sufficient conditions of the local uniform boundedness of motions perturbed with respect to the specified partial solution are obtained in this domain. The conditions found provide the local uniform boundedness of solutions irrespective of the forms of higher than the third order in the equations of perturbed motions. These conditions are obtained in the form of constraints for the coefficients of the normal form and, finally, for the original parameters of the system and the real and imaginary parts of the eigenvalues. To provide a clear interpretation of the results, special cases when all but two parameters are fixed are analysed. The domains of local uniform boundedness are constructed in the two-dimensional domains $F_0$ using a personal computer.

THE KINETIC energy, the force function and the equations of the angular momentum for a compound three-dimensional pendulum moving in the uniform field of gravity forces have been obtained in [1]. A generalized gyroscope in gimbals [2], a gyroscope in gimbals with constructive imperfections and a perfect gyroscope [3] are special cases of the compound three-dimensional pendulum. The local boundedness motions of a perfect gyroscope in gimbals with dissipative and accelerating forces has been examined in [4].

## 1. PARTIAL SOLUTION OF THE EQUATIONS OF MOTION. STATEMENT OF THE PROBLEM

Let us consider a massive gyroscope in gimbals with constructive imperfections in a Newtonian field of forces with the centre $S$. The outer ring $S_2$ (the frame) has the axis of rotation $l_2$ fixed on a stationary platform perpendicular to the direction $O_2S$. Here $O_2$ is a point on the $l_2$ axis whose

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position will be specified below. The $l_1$ axis of rotation of the inner ring $S_1$ (the mantle) is fixed in the body $S_2$. The $l_m$ axis, in general, is not the main or central one for the ring $S_m$, $m = 2, 1$. The gyroscope $S_0$ (the rotor) rotates about the $l_0$ axis of dynamical symmetry. This axis contains the centre of mass of the rotor and is fixed in the body $S_1$. The set $l_n$ $(n = 2, 1, 0)$, in general, does not contain any pair of orthogonal or intersecting axes. We assume that there is a moment of forces of viscous friction acting on the axis of the outer (or the inner) ring and there is an electromagnetic device on the axis of the inner (or the outer) ring which produces an accelerating moment acting in the direction of rotation of the ring and proportional to the angular velocity of its rotation [5, p. 182]. We will assume that there is no moment of friction forces about the axis of rotation of the gyroscope or that it is balanced by an electromagnetic moment applied to the rotor [6, p. 85].

The plane passing through the $l_m$ axis and parallel to the $l_{m-1}$ axis $(m = 2, 1)$ will be referred to as the basic plane of the ring $S_m$, while the basic plane of the ring $S_0$ will correspond to any plane which passes through the axis of symmetry $l_0$. We will introduce into the consideration the following orthogonal right-handed systems of coordinates. The system $O_2XYZ$ is attached to the stationary platform, the $O_2X$ axis is directed to the centre of gravitation $S$, and $O_2Y$ is in the direction of the stationary $l_2$ axis. The system $O_n\xi_n\eta_n\zeta_n$ is attached to the body $S_n$ $(n = 2, 1, 0)$, the $O_n\xi_n$ axis is in the direction of $l_n$, and $O_n\eta_n$ is likewise in the basic plane of the body $S_n$. The point $O_0$ is placed at the centre of mass of the rotor. We specify the point $O_m$ of the $l_m$ axis in such a way that the point $O_{m-1}$ is in the coordinate plane $O_m\eta_m\zeta_m$ $(m = 1, 2)$.

The angle between the $O_2X$ and $O_2\xi_2$ axes is equal to $\pi/2$, and the constant angle between the $O_m\xi_m$ and $O_{m-1}\xi_{m-1}$ axes is denoted by $\epsilon_m$ $(m = 2, 1)$. Let $X_n'$, $y_n'$ and $z_n'$ be the coordinates of the centre of mass of the body $S_n$ with respect to the system $O_n\xi_n\eta_n\zeta_n$ $(n = 2, 1, 0)$, and let $X_m'$, $Y_m'$ and $Z_m'$ be the components of the vector $O_mO_{m-1}$ with respect to the system $O_m\xi_m\eta_m\zeta_m$ $(m = 2, 1)$. We have $x_0' = y_0' = z_0' = X_2' = X_1' = 0$.

We will assume that the basic planes of the bodies in the initial positions are parallel to each other and to the direction $O_2S$. The current position of the system under consideration will be specified by the Cardan angles $\psi$, $\theta$ and $\varphi$. The angle $\psi$ of rotation of the frame $S_2$ specified in the plane perpendicular to the $l_2$ axis is the angle between the stationary plane $O_2XY$ and the basic plane $O_2\xi_2\eta_2$. The angle $\theta$ of rotation of the mantle $S_1$ (the angle $\varphi$ of rotation of the gyroscope $S_0$ itself) specified in the plane perpendicular to $l_1(l_0)$ is the angle between the basic planes $O_2\xi_2\eta_2$ and $O_1\xi_1\eta_1$ $(O_1\xi_1\eta_1$ and $O_0\xi_0\eta_0)$.

The equations of motion of the imperfect gyroscope in gimbals with dissipative and accelerating forces acting in the axes of the suspension rings have the form

$$\mathbf{p}^{\cdot} = -\partial H/\partial \mathbf{q} - \mathbf{F}\,\partial H/\partial \mathbf{p}, \qquad \mathbf{q}^{\cdot} = \partial H/\partial \mathbf{p} \tag{1.1}$$

Here $H = H(\cdot)$ is the Hamilton function, $\mathbf{p} = (\cdot)^{\mathrm{T}}$ are the generalized momenta corresponding to the coordinates $\mathbf{q} = (\psi, \theta, \varphi)^{\mathrm{T}}$, $\mathbf{F} = \operatorname{diag}(k_\psi, k_\theta, k_\varphi)$ is the matrix of the Rayleigh function

$$F = \tfrac{1}{2}k_\psi(\partial H/\partial p_\psi)^2 + \tfrac{1}{2}k_\theta(\partial H/\partial p_\theta)^2$$

$k_\psi > 0$ (or $k_\theta > 0$) is the coefficient of viscous friction acting on the axis of the frame (or of the mantle) and $k_\theta < 0$ (or $k_\psi < 0$), thus, $|k_\theta|$ (or $|k_\psi|$) is the steepness of the characteristic of the electromagnetic device which is located on the axis of the mantle (or of the frame) and produces an accelerating moment with $k_\psi k_\theta < 0$. For brevity, we will not write the Hamilton function $H$ here, noticing that its expression contains, apart from the magnitudes mentioned above, the following quantities. $B_{ij}$ and $A_{ij}$ $(i, j = 1, 2, 3, i \leqslant j)$ are the components of the tensor of inertia of the outer and inner rings, respectively, $A$ and $B$ are the components of the diagonal tensor of inertia of the gyroscope, $A_{11}^* = A_{11} + M_0(Y_1'^2 + Z_1'^2)$, $A_{22}^* = A_{22} + M_0Z_1'^2$, $A_{33}^* = A_{33} + M_0Y_1'^2$, $A_{12}^* = A_{12}$, $A_{13}^* = A_{13}$, $A_{23}^* = A_{23} - M_0Y_1'Z_1'$, $M_n$ is the mass of the body $S_n$ $(n = 2, 1, 0)$, $M = M_1 + M_0$, $\xi_1' = M^{-1}M_1x_1'$, $\eta_1' = M^{-1}(M_1y_1' + M_0Y_1')$, $\zeta_1' = M^{-1}(M_1z_1' + M_0Z_1')$, $g$ is the acceleration of the gravity force at the point $O_2$, and $R$ is the distance between the points $O_2$ and $S$.

Since the Hamiltonian $H$ does not contain the angle $\varphi$ of rotation of the gyroscope explicitly and the forces acting in the rotor axis are balanced, system (1.1) has the integral $p_\varphi = \text{const}$ and a reduced system with two degrees of freedom can be extracted from Eqs (1.1).

Let the conditions

$$M_2 y_2' + MY_2' + M\xi_1' \sin\epsilon_2 = 0, \quad \zeta_1' = 0$$
$$B_{23} + A_{12}\sin\epsilon_2 + (A-B)\sin\epsilon_2\cos\epsilon_1\sin\epsilon_1 = 0, \quad A_{23} = 0 \tag{1.2}$$

hold.

The equations of motion (1.1) of the imperfect gyroscope in gimbals then have the partial solution

$$p_\psi = A\omega'\cos\epsilon_2\cos\epsilon_1, \quad p_\theta = A\omega'\cos\epsilon_1$$
$$p_\varphi = A\omega', \quad \psi = \theta = \pi/2, \quad \varphi = \omega' t + \varphi_0 \tag{1.3}$$

According to this solution, the basic plane of the frame is perpendicular to the $O_2S$ direction, and the basic plane of the mantle is parallel to $O_2S$ and contains this direction if $Y_2' = 0$; the latter condition, in general, is not assumed in advance. The gyroscope rotates with an arbitrary constant angular velocity $\omega'$. The second condition in (1.2) means that the centre of mass of the "mantle-rotor" system lies in the basic plane of the mantle. The set of the first two conditions in (1.2) means that the centre of mass $P$ of the system of three bodies $S_2$, $S_1$ and $S_0$ lies in the plane which passes through the centre of gravitation $S$ and the $l_2$ axis while the system executes steady motions (1.3). The third and fourth conditions in (1.2) of the existence of solution (1.3) occur because of the assumption that the Newtonian field is central.

We will examine the sufficient conditions for local uniform boundedness [7, 8] of the perturbed motions of an imperfect gyroscope in gimbals with dissipative and accelerating forces acting in the axes of the suspension rings with respect to the variables $p_\psi$, $p_\theta$, $p_\varphi$, $\psi$ and $\theta$ for steady motions (1.2) when there are parametric perturbations of the constructive parameters of the system.

## 2. THE EQUATIONS OF PERTURBED MOTIONS OF THE REDUCED SYSTEM (1.1)

Let us find the equations of perturbed motions of the reduced system in the neighbourhood of the position of equilibrium

$$p_\psi = A\omega'\cos\epsilon_2\cos\epsilon_1, \quad p_\theta = A\omega'\cos\epsilon_1, \quad \psi = \theta = \pi/2, \tag{2.1}$$

which corresponds to the steady motions (1.3) of original system (1.1). We set $p_\psi = A\omega'\cos\epsilon_2\cos\epsilon_1 + p_1'$, $p_\theta = A\omega'\cos\epsilon_1 + p_2'$, $\psi = \pi/2 + q_1'$, $\theta = \pi/2 + q_2'$ and find the expansion of the Hamilton function of the reduced system in the neighbourhood of equilibrium (2.1) to within fourth-order terms with respect to the perturbations $p_m'$ and $q_m'$ $(m = 1, 2)$. We introduce the new dimensionless variables $p_m$ and $q_m$ $(m = 1, 2)$, the time $\tau$, the angular speed $\omega$, the coefficients $k_m$ and the parameters $b_i$, $a_{ij}$, $a_{ij}^*$ $(i, j = 1, 2, 3, i \leqslant j)$, $b$, $Y_2$, $Z_2$, $Y_1$, $Z_1$, $\xi_1$, $\eta_1$, $m$, $\delta$, $e$ by the formulae

$$\sigma_* = (g\,|M_2 z_2' + MZ_2' + M\eta_1'|/A)^{1/2}, \quad p_m' = A\sigma_* p_m$$
$$q_m' = q_m\ (m = 1, 2), \quad t = \sigma_*^{-1}\tau, \quad \omega' = \sigma_*\omega, \quad k_\psi = A\sigma_* k_1$$
$$k_\theta = A\sigma_* k_2, \quad b_i = \frac{B_{ii}}{A}, \quad a_{ij} = \frac{A_{ij}}{A}\ (a_{23} = 0)$$
$$a_{ij}^* = \frac{A_{ij}^*}{A}\ (i, j = 1, 2, 3, i \leqslant j), \quad b = \frac{B}{A}, \quad Y_2' = \left(\frac{A}{M}\right)^{1/2} Y_2 \tag{2.2}$$
$$Z_2' = \left(\frac{A}{M}\right)^{1/2} Z_2, \quad Y_1' = \left(\frac{A}{M_0}\right)^{1/2} Y_1, \quad Z_1' = \left(\frac{A}{M_0}\right)^{1/2} Z_1$$
$$\xi_1' = \left(\frac{A}{M}\right)^{1/2} \xi_1, \quad \eta_1' = \left(\frac{A}{M}\right)^{1/2} \eta_1, \quad m = \frac{gM\eta_1'}{A\sigma_*^2}$$
$$\delta = \frac{g}{R\sigma_*^2}, \quad e = \operatorname{sign}(M_2 z_2' + MZ_2' + M\eta_1')$$

The dimensionless parameters $\epsilon_1$ and $\epsilon_2$ remain the same. It turns out that there are 23 independent dimensionless parameters.

We obtain the expansion of the Hamilton function of the reduced system expressed in terms of the dimensionless variables as

$$H = H_2 + H_3 + H_4 + \ldots$$
$$H_n = \sum_{|\nu|=n} h_{\nu_1\nu_2\nu_3\nu_4} p_1^{\nu_1} p_2^{\nu_2} q_1^{\nu_3} q_2^{\nu_4} \quad (n = 2, 3, 4, \ldots), \tag{2.3}$$

$\nu_1, \ldots, \nu_4$ are non-negative integers, $|\nu| = \nu_1 + \ldots + \nu_4$. The non-zero coefficients $h_{\nu_1\nu_2\nu_3\nu_4}$ of the forms $H_2$, $H_3$ and $H_4$ are

$$\begin{aligned}
&2h_{2000} = \Theta_0 D_0,\ h_{1100} = -\Phi_0 D_0,\ h_{1001} = -\pi_0 \Theta_0 D_0,\ 2h_{0200} = \Psi_0 D_0,\\
&h_{0101} = \pi_0 \Phi_0 D_0,\ 2h_{0020} = e + 3\delta h_{20},\ h_{0011} = m\cos\epsilon_2 + 3\delta h_{11},\ 2h_{0002} =\\
&= \pi_0^2 \Theta_0 D_0 + m + 3\delta h_{02},\ 2h_{2001} = -\Theta_0 D_1,\ h_{1101} = -\Phi_1 D_0 + \Phi_0 D_1,\ h_{1002} = \pi_0 \Theta_0 D_1,\\
&2h_{0201} = \Psi_1 D_0 - \Psi_0 D_1,\ h_{0102} = \pi_0(\Phi_1 D_0 - \Phi_0 D_1),\ h_{0021} = 3\delta\cos\epsilon_2 h_{12},\\
&2h_{0012} = 3\delta h_{12},\ 2h_{0003} = -\pi_0^2 \Theta_0 D_1,\ 2h_{2002} = \Theta_0 D_2,\ h_{1102} = -\Phi_2 D_0 + \Phi_1 D_1 -\\
&- \Phi_0 D_2,\ h_{1003} = \pi_0 \Theta_0 (1/6 D_0 - D_2),\ 2h_{0202} = \Psi_2 D_0 - \Psi_1 D_1 + \Psi_0 D_2,\\
&2h_{0103} = \pi_0(-{}^1\!/_3\, \Phi_0 + 2\Phi_2) D_0 + 2\pi_0(-\Phi_1 D_1 + \Phi_0 D_2),\ 2h_{0040} = -1/12 e - \delta h_{20},\\
&h_{0031} = -{}^1\!/_6\, m\cos\epsilon_2 - 2\delta h_{11},\ 2h_{0022} = -{}^1\!/_2 m + 3\delta h_{22},\ h_{0013} = -{}^1\!/_6\, m\cos\epsilon_2 -\\
&- 2\delta(h_{11} + {}^3\!/_4 a_{13}\sin\epsilon_2)\\
&2h_{0004} = \pi_0^2 \Theta_0(-{}^1\!/_3 D_0 + D_2) - 1/12 m - \delta h_{02}
\end{aligned} \tag{2.4}$$

Here

$$\begin{aligned}
&h_{20} = b_2 - b_3 + a_{11}\sin^2\epsilon_2 - a_{22} + a_{33}\cos^2\epsilon_2 - a_{13}\sin 2\epsilon_2 + (b-1)(\sin^2\epsilon_1 - \sin^2\epsilon_2\cos^2\epsilon_1)\\
&h_{11} = (-a_{22} + a_{33})\cos\epsilon_2 - a_{13}\sin\epsilon_2 + (b-1)\cos\epsilon_2 \sin^2\epsilon_1\\
&h_{02} = -a_{22} + a_{33} + (b-1)\sin^2\epsilon_1,\ h_{12} = \sin\epsilon_2(-a_{12} + (b-1)\cos\epsilon_1\sin\epsilon_1)\\
&h_{22} = (a_{22} - a_{33} - (b-1)\sin^2\epsilon_1)(1 + \cos^2\epsilon_2) + {}^1\!/_2 a_{13}\sin 2\epsilon_1\\
&\pi_0 = \omega\sin\epsilon_2\sin\epsilon_1,\ \Theta_0 = a_{11}^* + b\sin^2\epsilon_1\\
&\Psi_0 = b_1 + a_{11}^*\cos^2\epsilon_2 + a_{33}^*\sin^2\epsilon_2 + a_{13}^*\sin 2\epsilon_2 + b(1 - \cos^2\epsilon_2\cos^2\epsilon_1) + Y_2^2 + Z_2^2 +\\
&+ 2Y_2\xi_1\sin\epsilon_2 + 2Z_2\eta_1\\
&\Phi_0 = a_{11}^*\cos\epsilon_2 + a_{13}^*\sin\epsilon_2 + b\cos\epsilon_2\sin^2\epsilon_1 + Z_2\eta_1\cos\epsilon_2,\ \Delta_0 = \Theta_0\Psi_0 - \Phi_0^2,\ D_0 = \Delta_0^{-1}\\
&\Psi_1 = a_{12}^*\sin 2\epsilon_2 + 2a_{23}^*\sin^2\epsilon_2 - {}^1\!/_2 b\sin 2\epsilon_2\sin 2\epsilon_1 - 2Y_2\eta_1\cos\epsilon_2\\
&\Phi_1 = a_{12}^*\sin\epsilon_2 - {}^1\!/_2 b\sin\epsilon_2\sin 2\epsilon_1 - Y_2\eta_1\\
&\Delta_1 = \Theta_0\Psi_1 - 2\Phi_0\Phi_1,\ D_1 = \Delta_1\Delta_0^{-2}\\
&\Psi_2 = (a_{22}^* - a_{33}^*)\sin^2\epsilon_2 - {}^1\!/_2 a_{13}^*\sin 2\epsilon_2 - b\sin^2\epsilon_2\sin^2\epsilon_1 - Z_2\eta_1,\\
&\Phi_2 = -{}^1\!/_2(a_{13}^*\sin\epsilon_2 + Z_2\eta_1\cos\epsilon_2),\ \Delta_2 = \Theta_0\Psi_2 - \Phi_1^2 - 2\Phi_0\Phi_2,\ D_2 = (\Delta_1^2 - \Delta_0\Delta_2)\Delta_0^{-3}
\end{aligned} \tag{2.5}$$

The equations of perturbed motions of the reduced system in the neighbourhood of equilibrium (2.1) for the dimensionless variables defined by (2.2) have the form

$$dp_m/d\tau = -\partial H/\partial q_m - k_m\,\partial H/\partial p_m,\ dq_m/d\tau = \partial H/\partial p_m \ (m = 1, 2) \tag{2.6}$$

*Remarks.* 2.1. When considering the approximate investigation of the central Newtonian field it is assumed that the distance $R$ should be much greater than the dimensions of the gyroscope in gimbals. For this reason, the dimensionless parameter $\delta$ is small. The limiting case $\delta = 0$ corresponds to the uniform field of gravity forces.

2.2. If the centre of mass $P$ of the system of three bodies $S_2$, $S_1$ and $S_0$ and the centre of gravitation $S$ are located on one side (or on different sides) of the basic plane of the frame $S_2$ when steady motions (1.3) are occurring, we obtain $e = 1$ (or $e = -1$).

## 3. THE CHARACTERISTIC EQUATION. ASSUMPTIONS ON THE PROPERTIES OF THE EIGENVALUES

Consider the domain of admissible values of the parameters

$$F_0 = \{\mathbf{c} = (\omega, k_1, k_2, b_1, b_2, b_3, a_{11}, a_{12}, a_{13}, a_{22}, a_{33}, b, \epsilon_2, \epsilon_1, Y_2, Z_2, Y_1, Z_1, \xi_1, \eta_1, m, \delta):\ k_1 > 0, k_2 < 0\ (\text{or}\ \ k_1 < 0, k_2 > 0), b_1 + b_2 > b_3, b_2 + b_3 > b_1,\ b_3 + b_1 > b_2, a_{11} + a_{22} > a_{33}, a_{22} + a_{33} > a_{11}, a_{33} + a_{11} > a_{22}, a_{11}a_{22} - a_{12}^2 > 0, a_{11}a_{22}a_{33} - a_{22}a_{13}^2 - a_{33}a_{12}^2 > 0,\ b > ½,\ \epsilon_2 \in [0, \pi), \epsilon_1 \in [0, \pi),\ \eta_1 m > 0, \Delta_0 > 0, \delta \text{ is a small positive quantity}\}, \text{ and } e = \pm 1.$$

The characteristic equation of system (2.6) has the form

$$\lambda^4 + P_1\lambda^3 + (P_2 + Q_2)\lambda^2 + P_3\lambda + Q_4 = 0 \tag{3.1}$$

If we calculate the coefficients of system (3.1) as in [9], we obtain

$$\begin{aligned} &P_1 = (k_1\Theta_0 + k_2\Psi_0)D_0,\ \ P_2 = k_1k_2D_0 \\ &Q_2 = (\pi_0^2 + \Theta_0(e + 3\delta h_{20}) - 2\Phi_0(m\cos\epsilon_2 + 3\delta h_{11}) + \Psi_0(m + 3\delta h_{02}))D_0, \\ &P_3 = (k_1(m + 3\delta h_{02}) + k_2(e + 3\delta h_{20}))D_0, \\ &Q_4 = ((e + 3\delta h_{20})(m + 3\delta h_{02}) - (m\cos\epsilon_2 + 3\delta h_{11})^2)D_0 \end{aligned} \tag{3.2}$$

The substitution of formulae (2.5) into (3.2) yields the final expression for the coefficients of the characteristic polynomial in terms of the parameters $\mathbf{c}\in F_0$, $e = \pm 1$.

Suppose that Eq. (3.1) has two pairs of complex-conjugate roots. From the Vièta formulae it follows that, under the assumption made above, the condition $Q_4 > 0$ is necessary. This condition is satisfied, according to Remark 2.1, when $|m| \leqslant 1$. Using the restrictions having the form of strict inequalities on the coefficients of the characteristic polynomial, the decomposition of the domain $F_0$ into the parts $N_l$, $l = 0, 4, 2$, has been constructed so that the $l$ roots of Eq. (3.1) are located in the half-plane $\mathrm{Re}\lambda > 0$ and $4 - l$ roots are located in the half-plane $\mathrm{Re}\lambda < 0$ if $\mathbf{c}\in N_l$.

If $\mathbf{c}\in N_0$, the uniform rotations (1.3) of the imperfect gyroscope in gimbals are asymptotically stable [10] with respect to the variables $p_\psi$, $p_\theta$, $p_\varphi$, $\psi$ and $\theta$ under parametric perturbations of the constructive parameters. If $\mathbf{c}\in N_4 \cup N_2$, the steady motions (1.3) are unstable [10]. But in reality, the initial perturbations, generally speaking, cannot be taken from a neighbourhood of the unperturbed motion as small as desired specified in advance. Then, the existence of roots of the defining equation with only negative real parts is not a criterion of stability, while the existence of roots with positive real parts is not an indication of instability regardless of the non-linear terms in the equations of perturbed motions [11]. The problem of the qualitative behaviour of the solutions of Eqs (2.6) of perturbed motions as a function of the non-linear terms remains unsolved.

In order to examine the local uniform boundedness of solutions of system (2.6) with respect to the origin of coordinates we will introduce additional assumptions on the properties of the eigenvalues.†

*Assumption 1.* All the roots of the characteristic equations (3.1), (3.2) have real parts of small magnitude.

†The definition of local uniform boundedness used in [7, 8] for the solutions of an autonomous system of the form (2.6) was stated in: BELIKOV S. A., Local boundedness of solutions of a fourth-order autonomous system with small positive real parts of the eigenvalues, Leningrad, 1987. Unpublished paper. Deposited in VINITI 26.03.87, 2206–B87.

Denote the roots of Eqs (3.1) and (3.2) by $\mu\alpha_m \pm i\beta_m$ $(m = 1, 2)$ where $\mu > 0$ is a small parameter, $\alpha_m = O(1)$, and $\beta_m > 0$. We will assume, to fix our ideas, that $\beta_1 > \beta_2$.

Consider the hypersurface of codimensions 2

$$P_0 = \{\mathbf{c}: \mathbf{c} \in F_0,\ P_1(\mathbf{c}) = 0,\ P_3(\mathbf{c}) = 0\},\quad e = \pm 1$$

and the domain

$$G = \{\mathbf{c}: \mathbf{c} \in F_0,\ P_2(\mathbf{c}) + Q_2(\mathbf{c}) > 2(Q_4(\mathbf{c}))^{1/2}\},\quad e = \pm 1$$

We notice that for $\mathbf{c} \in P_0 \cap G$ all the eigenvalues are pure imaginary ($\mu = 0$). For Assumption 1 to be satisfied the inclusion $\mathbf{c} \in P_0^0$ is necessary and $\mathbf{c} \in P_0^0 \cap G$ is sufficient. Here $P_0^0$ is a small neighbourhood of the surface $P_0$ such that for any $\mathbf{c} \in P_0^0$ the values of $|P_1(\mathbf{c})|$ and $|P_3(\mathbf{c})|$ are small and $\mu = O(|P_1|, |P_2|)$.

Let us find the $(N+1)$th-order values of the off-resonance mistuning $\epsilon_{1,N} = \beta_1 - N\beta_2$ between the imaginary parts $\beta_1$ and $\beta_2$ of the eigenvalues, $N = 1, 2, 3$.

*Assumption 2.* The absolute values of the second- to fourth-order off-resonance mistuning between the imaginary parts of the roots of characteristic Eqs (3.1) and (3.2) are not small.

Consider the hypersurfaces of codimensions 1

$$\begin{aligned}
R_2 &= \{\mathbf{c}: \mathbf{c} \in F_0,\ P_2(\mathbf{c}) + Q_2(\mathbf{c}) = 2(Q_4(\mathbf{c}))^{1/2}\}\\
R_3 &= \{\mathbf{c}: \mathbf{c} \in F_0,\ P_2(\mathbf{c}) + Q_2(\mathbf{c}) = 5/2(Q_4(\mathbf{c}))^{1/2}\}\\
R_4 &= \{\mathbf{c}: \mathbf{c} \in F_0,\ P_2(\mathbf{c}) + Q_2(\mathbf{c}) = 10/3(Q_4(\mathbf{c}))^{1/2}\},\quad e = \pm 1
\end{aligned}$$

Note that $R_2$ is the boundary of the domain $G$ while $R_3 \subset G$ and $R_4 \subset G$. If the inclusion $\mathbf{c} \in P_0^0 \cap R_{n+1}$ holds, the absolute values of the off-resonance mistuning $\epsilon_{1,N}(\mathbf{c})$ $(N = 1, 2, 3)$ is small. Because of Assumption 2, we exclude from consideration a small neighbourhood $R_{N+1}^0$ of the surface $R_{N+1}$, such that $(\mathbf{c} \in P_0^0 \cap G \backslash R_{N+1}^0)\,|\epsilon_{1,N}(\mathbf{c})| > \sqrt{\mu}$, $N = 1, 2, 3$.

## 4. NORMALIZATION OF THE EQUATIONS OF PERTURBED MOTIONS AND THE SUFFICIENT CONDITIONS OF THE LOCAL BOUNDEDNESS

Let the inclusion $\mathbf{c} \in P_0^0 \cap G \backslash R_2^0$ hold. The linear change of the variables $\mathbf{z} = \mathbf{S}\mathbf{x}$, $\mathbf{z} = (p_1, p_2, q_1, q_2)^{\mathrm{T}}$, $\mathbf{S} = \|s_{kl}\|_{k,l=1}^4$, $\det \mathbf{S} \neq 0$, $\mathbf{x} = (x_1, x_2, x_3, x_4)^{\mathrm{T}}$ transforms the matrix $\mathbf{B}$ of linearized system (2.6) into the real Jordan form. We shall write down the elements of the matrix $S$. If we put

$$\begin{aligned}
r_1 &= k_1(\mu\alpha_1\Phi_0 + \pi_0) + \Phi_0(e + 3\delta h_{20}) - \Psi_0(m\cos\epsilon_2 + 3\delta h_{11}),\quad i_1 = k_1\beta_1\Phi_0,\\
r_2 &= ((\mu\alpha_1)^2 - \beta_1^2)\Delta_0 + \mu\alpha_1(k_1\Theta_0 - \pi_0\Phi_0) + \Theta_0(e + 3\delta h_{20}) - \Phi_0(m\cos\epsilon_2 + 3\delta h_{11}) \qquad (4.1)\\
i_2 &= 2\mu\alpha_1\beta_1\Delta_0 + \beta_1(k_1\Theta_0 - \pi_0\Phi_0),\quad r_3 = ((\mu\alpha_1)^2 - \beta_1^2)\Phi_0 + \mu\alpha_1\pi_0,\\
i_3 &= \beta_1(2\mu\alpha_1\Phi_0 + \pi_0),\quad r_4 = ((\mu\alpha_1)^2 - \beta_1^2)\Psi_0 + k_1\mu\alpha_1,\quad i_4 = \beta_1(2\mu\alpha_1\Psi_0 + k_1),\\
g(\alpha_1, \beta_1) &= |\mu\alpha_1(r_1 i_3 - i_1 r_3 - r_2 i_4 + i_2 r_4) - \beta_1(r_1 r_3 + i_1 i_3 - r_2 r_4 - i_2 i_4) -\\
&\quad - (\mu\alpha_1 i_1 + \beta_1 r_1)(m\cos\epsilon_2 + 3\delta h_{11}) + (\mu\alpha_1 i_2 + \beta_1 r_2 + \pi_0 i_3)(e + 3\delta h_{20})|^{-1/2}
\end{aligned}$$

we have

$$\begin{aligned}
s_{11} &= (-\mu\alpha_1 r_1 + \beta_1 i_1 - \pi_0(e + 3\delta h_{20}))g(\alpha_1, \beta_1)\\
s_{21} &= (-\mu\alpha_1 r_2 + \beta_1 i_2)g(\alpha_1, \beta_1),\quad s_{31} = (r_3 + m\cos\epsilon_2 + 3\delta h_{11})g(\alpha_1, \beta_1), \qquad (4.2)\\
s_{41} &= -(r_4 + e + 3\delta h_{20})g(\alpha_1, \beta_1),\quad s_{13} = (\mu\alpha_1 i_1 + \beta_1 r_1)g(\alpha_1, \beta_1)\\
s_{23} &= (\mu\alpha_1 i_2 + \beta_1 r_2)g(\alpha_1, \beta_1),\quad s_{33} = -i_3 g(\alpha_1, \beta_1),\quad s_{43} = i_4 g(\alpha_1, \beta_1)
\end{aligned}$$

The formulae for $s_{k2}$ and $s_{k4}$ $(k = 1, \ldots, 4)$ are obtained from the expressions for $s_{k1}$ and $s_{k3}$ $(k = 1, \ldots, 4)$, respectively, by making the changes $\alpha_1 \to \alpha_2$ and $\beta_1 \to \beta_2$ in (4.1) and (4.2). Substitution of (4.1) and (2.5) into (4.2) yields the final expressions for the elements of the matrix $\mathbf{S}$ in terms of the original dimensionless parameters $\mathbf{c} \in P_0^0 \cap G \backslash R_2^0$ and the quantities $\alpha_m$ and $\beta_m$ $(m = 1, 2)$.

Suppose the inclusion

$$\mathbf{c} \in P_0^0 \cap G \backslash (R_2^0 \cup R_3^0 \cup R_4^0) \tag{4.3}$$

holds.

As a result of the normalizing transformation constructed by the Deprit–Hori–Kamel method, the system of ordinary differential equations $x$ obtained from (2.6) by the change of variables $\mathbf{z} = \mathbf{Sx}$ can be reduced to the normal form which is continuous at the point $\mu = 0$ with respect to the parameter $\mu$ to within third-order terms with respect to the variables transformed (see the paper mentioned in the previous footnote and [4]). We denote the coefficients of the continuous normal form by $\varphi_{m,10}$ and $\varphi_{m,01}$ $(m = 1, 2)$ [4]. The expressions for these coefficients in terms of $h_{\nu_1\nu_2\nu_3\nu_4}$ (2.4), (2.5), $|\nu| = 3, 4$, $k_m$, $\alpha_m$, $\beta_m$ $(m = 1, 2)$ and the elements of the matrices $\mathbf{S}$ and $\mathbf{S}^{-1}$ have been found using the analytic results of [9] and of the paper mentioned in the previous footnote. For brevity the expressions for the coefficients $\varphi_{m,10}$ and $\varphi_{m,01}$ $(m = 1, 2)$ obtained are not written here.

The following theorem on local boundedness holds.

*Theorem.* Suppose the parameters $\mathbf{c}$ satisfy inclusion (4.3), and $e = \pm 1$. Let the coefficients $\varphi_{m,10}$ and $\varphi_{m,01}$ $(m = 1, 2)$ of the continuous normal form, with the set $\mathbf{c}$ fixed and $e = \pm 1$, satisfy the following conditions:

1. $\varphi_{1,10} < 0$ and $\varphi_{2,01} < 0$ if $\varphi_{1,10}\varphi_{2,01} \leqslant 0$ or $\varphi_{1,01} \leqslant 0$ and $\varphi_{2,10} \leqslant 0$;
2. $\varphi_{1,01} < 0$, $\varphi_{2,10} < 0$ and $\varphi_{1,10}\varphi_{2,01} > \varphi_{1,01}\varphi_{2,10}$ if $\varphi_{1,01} > 0$ and $\varphi_{2,10} > 0$.

Then the solutions of Eqs (2.6) of perturbed motions are locally uniformly bounded with respect to the origin of coordinates $p_m = q_m = 0$ $(m = 1, 2)$ irrespective of the forms of the order higher than three.

*Remarks.* 4.1. The zero solution of Eqs (2.6) of perturbed motions describes the position of equilibrium (2.1) of the reduced system.

4.2. The assumptions of the theorem contain restrictions in the form of inequalities for the coefficients $\varphi_{m,10}$ and $\varphi_{m,01}$ $(m = 1, 2)$ of the continuous normal form. Note that, taking account of formulae (3.2), (4.2), (4.1) and (2.5), quantities $\alpha_m$ and $\beta_m$ $(m = 1, 2)$ and the elements of the matrices $\mathbf{S}$ and $\mathbf{S}^{-1}$ are expressed in terms of the parameters $\mathbf{c}$ (4.3), $e = \pm 1$ by the use of well-known algebraic relations. If we use the expressions mentioned above, which give the coefficients of the normal form, we can interpret the sufficient conditions for the local uniform boundedness obtained in the theorem as the conditions imposed on the original construction parameters of the imperfect gyroscope in gimbals.

The theorem on the continuous dependence of the solutions of system (1.1) on the parameters and the assumptions of the theorem stated above imply the following result. The perturbed motions of the imperfect gyroscope in gimbals with dissipative and accelerating forces acting on the axes of the suspension rings are locally uniformly bounded in $p_\psi$, $p_\theta$, $p_\varphi$, $\psi$ and $\theta$ with respect to the steady motions (1.3) under parametric disturbances of the construction parameters of the system.

## 5. INTERPRETATION OF THE CONDITIONS FOR LOCAL BOUNDEDNESS IN THE SPECIAL CASE

The above model of the imperfect gyroscope in gimbals contains many construction parameters. To provide an illustrative interpretation of the sufficient conditions for local uniform boundedness as the restrictions on the dimensionless parameters $\mathbf{c} \in F_0$, consider the sets $\{\mathbf{c}\} \subset P_0^0 \subset F_0$, $e = \pm 1$ in which values of all but two parameters, for instance, $b$ and $\omega$, are fixed. A FORTRAN program was written and debugged. It can interpret the conditions for local uniform boundedness for any such a set and compute an initial estimate $q$ and a current estimate $p$ of the local boundedness of solutions of the normal form.

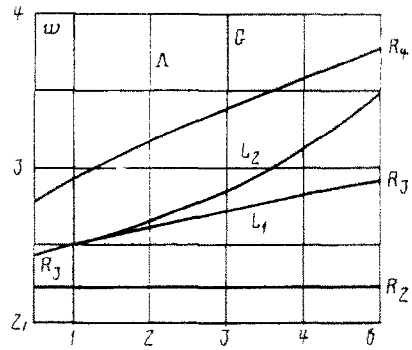


FIG. 1.

The interpretation of results computed for $k_1 = 15/8$, $k_2 = -3/2$, $b_1 = 1/4$, $b_2 = 1/2$, $b_3 = 3/8$, $a_{11} = 1$, $a_{12} = 1/4$, $a_{13} = \sqrt{3}/8$, $a_{22} = 1/2$, $a_{33} = 3/4$, $\epsilon_2 = \epsilon_1 = \pi/3$, $Y_2 = -1/4$, $Z_2 = 1/4$, $Y_1 = 1/2$, $Z_1 = 0$, $\xi_1 = \sqrt{3}/4$, $\eta_1 = 1/4$, $m = 4/5$, $\delta = 10^{-5}$, $e = +1$ is shown in Fig. 1. The analysis was carried out at the intersection of the rectangle $\{b \in [1/2; 10]\} \times \{\omega \in [2; 10]\}$ and the domain $G$ located above the curve $R_2$. The domain of local uniform boundedness is the domain $\Lambda$ located at the lower left of the curve $L_1$ and the upper left of the curve $L_2$. Some small neighbourhoods of the curves $R_2$, $R_3$ and $R_4$ are excluded from the domain $\Lambda$.

Calculations were also carried out for many other sets of the parameters mentioned above. We will merely outline the results obtained. For $k_1 > 0$, $k_2 < 0$, $e = \pm 1$, the domain $\Lambda$ in which the sufficient conditions of local uniform boundedness are satisfied, was found in all the cases examined. When $e = -1$ the domain $\Lambda$ was found rather seldom and for quite large values of $\omega$. For instance, the case $k_1 = -15/8$, $k_2 = 3/2$, $b_1 = 2$, $b_2 = 1$, $b_3 = 3/2$, $a_{11} = 3/2$, $a_{12} = -1/2$, $a_{13} = \sqrt{3}/2$, $a_{22} = 2$, $a_{33} = 1$, $\epsilon_2 = \epsilon_1 = \pi/3$, $Y_2 = \sqrt{3}/2$, $Z_2 = 1$, $Y_1 = \sqrt{2}/2$, $Z_1 = 0$, $\xi_1 = -13/12$, $\eta_1 = -1$, $m = -4/5$, $\delta = 10^{-5}$, $e = -1$ is characterized similarly.

## REFERENCES


1. KHARLAMOV P. V., A composite space pendulum. *Mekh. Tverdogo Tela, Naukova Dumka, Kiev* **4**, 73–82, 1972.
2. KONOSOVICH B. I., The rate of drift of the rotor axis in the generalized problem of a gyroscope in gimbals. *Mekh. Tverdogo Tela, Naukova Dumka, Kiev* **4**, 92–93, 1972.
3. KHARLAMOV M. P., Phase topology of a problem of the motion of a gyroscope. *Mekh. Tverdogo Tela, Naukova Dumka, Kiev* **13**, 14–23, 1981.
4. BELIKOV S. A., The local boundedness of perturbed motions of a gyroscope in gimbals with dissipative and accelerating forces. *Prikl. Mat. Mekh.* **54**, 958–965, 1990.
5. MERKIN D. R., *Introduction to the Theory of the Stability of Motion*. Nauka, Moscow, 1971.
6. LUNTS Ya. L., *Introduction to the Theory of Gyroscopes*. Nauka, Moscow, 1972.
7. ROUCHE N., HABETS P. and LALOY M., *Stability Theory by Lyapunov's Direct Method*. Mir, Moscow, 1980.
8. BELIKOV S. A., Local boundedness of solutions of a system of ordinary differential equations with respect to a set *Diff. Uravn.* **23**, 568–577, 1987.
9. BELIKOV S. A., Local boundedness of solutions of the autonomous Hamilton equations with two degrees of freedom with dissipative forces. *Proceedings of the Fifth All-Union Chetayev Conf. on Analytical Mechanics, Stability and Control of Motion*, Kazan. Aviats. Inst., Kazan, 1987.
10. LYAPUNOV A. M., *General Problem on the Stability of Motion*. Gostekhizdat, Moscow and Leningrad, 1950.
11. KAMENKOV G. V., *Stability and Oscillations of Non-linear Systems*. Nauka, Moscow, 1972.


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